

Finite volume effects in chiral perturbation theory

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Outline

- Introduction
- Recent results in the ϵ -regime
- Recent results in the p -regime
- Lüscher's formula and extensions thereof
- Summary

Introduction

CHPT: expansion in m_{ql}/Λ and p/Λ

In finite volume the momentum is quantized:

$$p = \frac{2\pi}{L}n$$

Condition of applicability of CHPT:

$$m_{ql} \ll \Lambda \quad \text{and} \quad \frac{2\pi}{L} \ll \Lambda$$

$$\Lambda \sim 4\pi F_\pi \quad \Rightarrow \quad 2LF_\pi \gg 1$$

Once this condition is respected we still have two different physical situations

$$LM_\pi \lesssim 1 \quad \Rightarrow \quad \epsilon\text{-regime} \quad M_\pi \sim \frac{1}{L^2} \sim O(\epsilon^2)$$

$$LM_\pi \gg 1 \quad \Rightarrow \quad p\text{-regime} \quad M_\pi \sim \frac{1}{L} \sim O(p)$$

Axial-charge correlator

$$Q_k^A(t) = -i \int d^3x A_0^k(x) \quad \langle Q_i^A(t) Q_k^A(0) \rangle \equiv \delta_{ik} L^3 \Gamma_A(t)$$

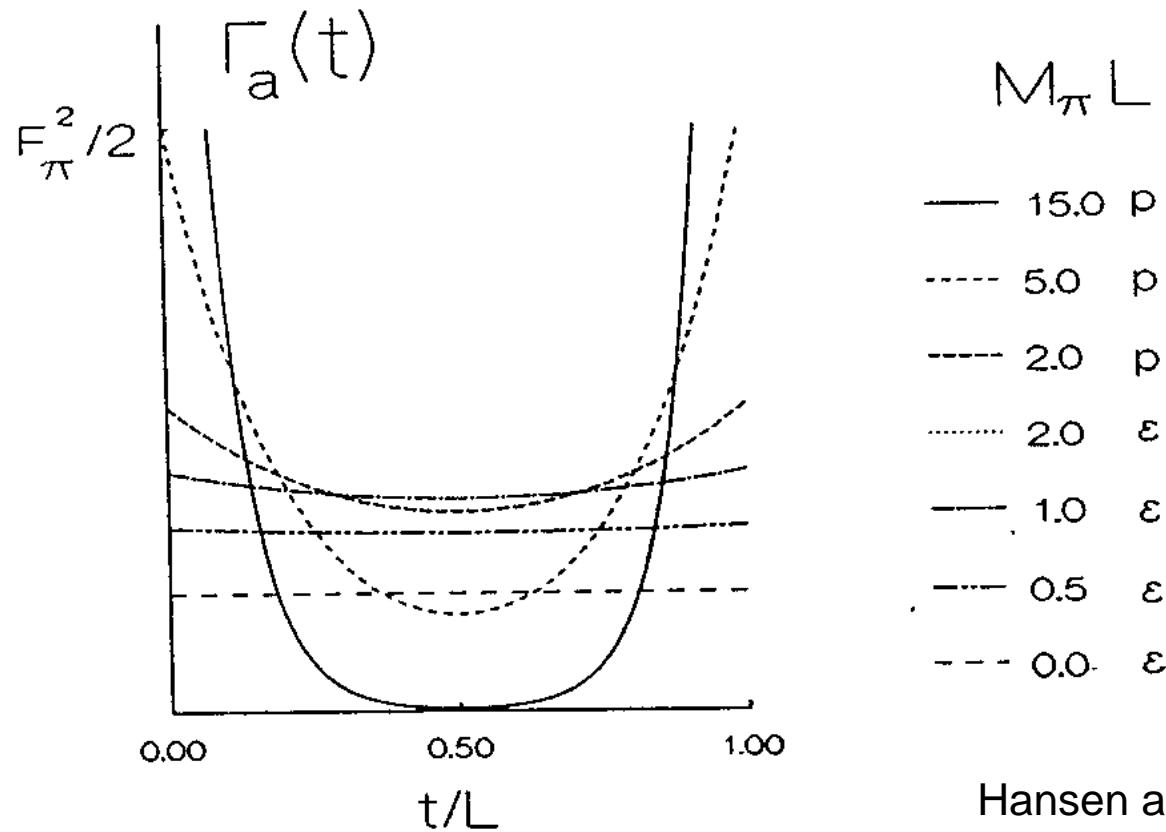
$$L = \infty \quad \Gamma_A(t) = \frac{M_\pi F_\pi^2}{2} e^{-M_\pi t} [1 + O(p^4)]$$

$$LM_\pi \gg 1 \quad \Gamma_A(t) = \frac{M_\pi^V F_{\pi,V}^2}{2} \frac{\cosh(M_\pi^V(L_t/2 - t))}{\sinh(M_\pi^V L_t/2)} [1 + O(p^4)]$$

$$LM_\pi \lesssim 1 \quad \Gamma_A(t) = \frac{F^2}{L_t} \left[\frac{\gamma_a}{F^2} + \frac{L_t}{F_\pi^2 L^3} g_1(u) h_1(t/L_t) + O(\epsilon^4) \right]$$

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Hansen and Leutwyler 91

p - or ϵ -regime?

Two alternatives:

- Chiral limit on the lattice $\Rightarrow \epsilon$ -regime
(unless one can simulate enormous volumes)
 \Rightarrow Rely on CHPT to relate unphysical observables to physical quantities
- $M_\pi > M_\pi^{\text{phys}}$: choose $L \gg 1/M_\pi$, $\Rightarrow p$ -regime
(e.g. $M_\pi = 300$ MeV, $L = 2$ fm, $M_\pi L \sim 3$)
 \Rightarrow Rely on CHPT to make the chiral and the large volume extrapolation

The ϵ -regime is expensive!

- In the chiral limit the spectrum of the Dirac operator becomes dense at the origin
- The Dirac operator is ill-conditioned
- Large statistical fluctuations appear in correlation functions

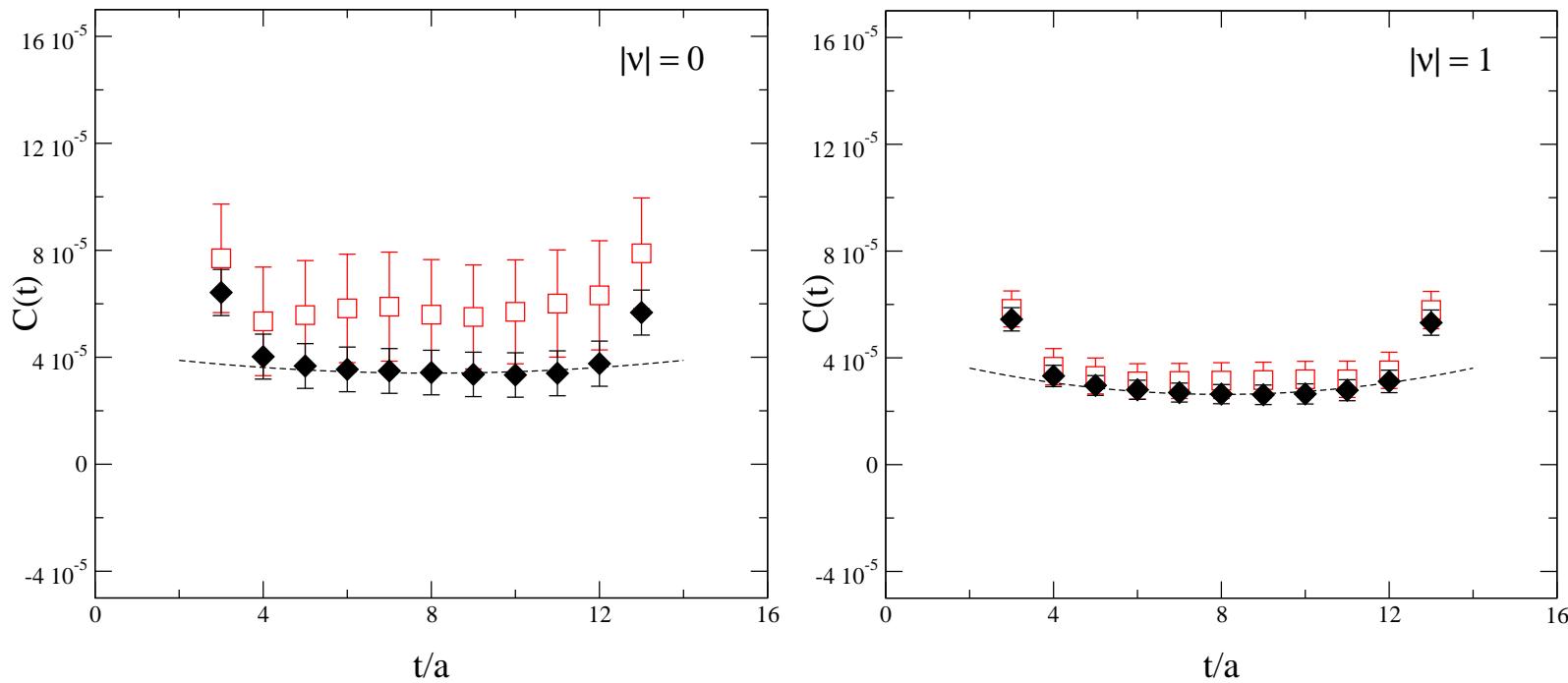
Bietenholz et al. (03), Giusti et al. (04)

- Efficient numerical tools are needed
“low mode preconditioning” and “low mode averaging”
Giusti et al. (02, 04), DeGrand and Schaefer (04)

see H. Wittig’s talk at this conference
this slide is taken almost verbatim from there – thank you!

Recent results in the ϵ -regime

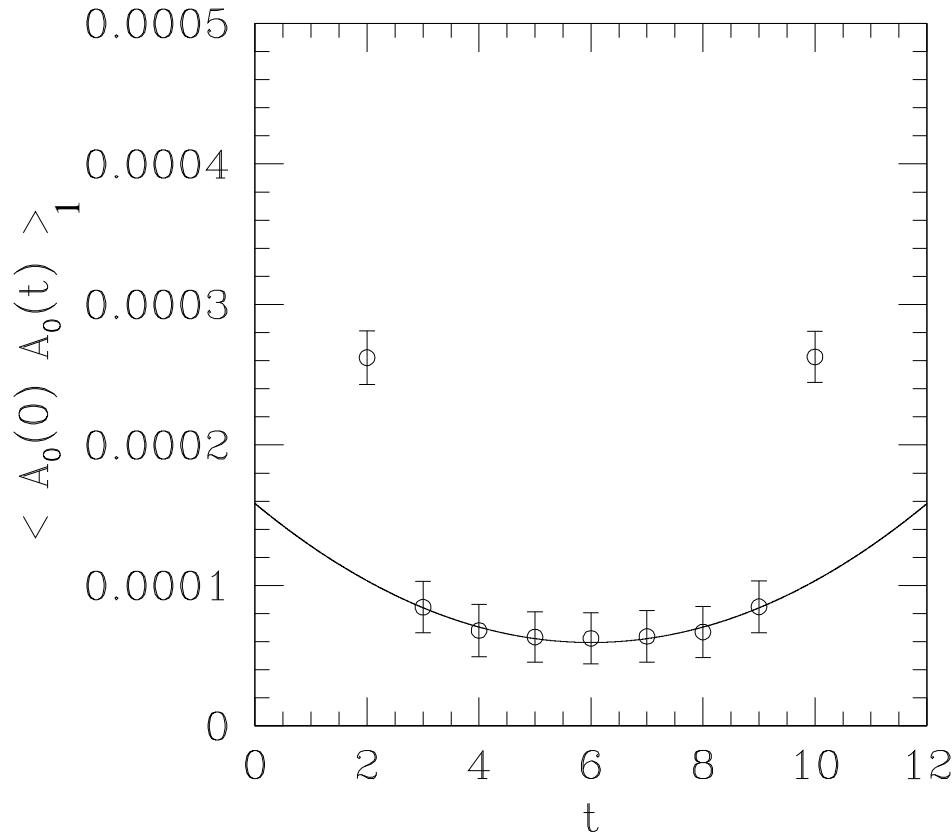
Left-left correlator:



from Giusti, Hernández, Laine, Weisz and Wittig (04)

Recent results in the ϵ -regime

Axial–axial correlator:



from Bietenholz et al. (03)

Recent results in the ϵ -regime

Determination of F_π in the chiral limit (quenched)

$$F \simeq 130 \text{ MeV}$$

Bietenholz et al.

$$F = 102 \pm 4 \text{ MeV}$$

Giusti et al., $[\epsilon]$

$$F = 104 \pm 2 \text{ MeV}$$

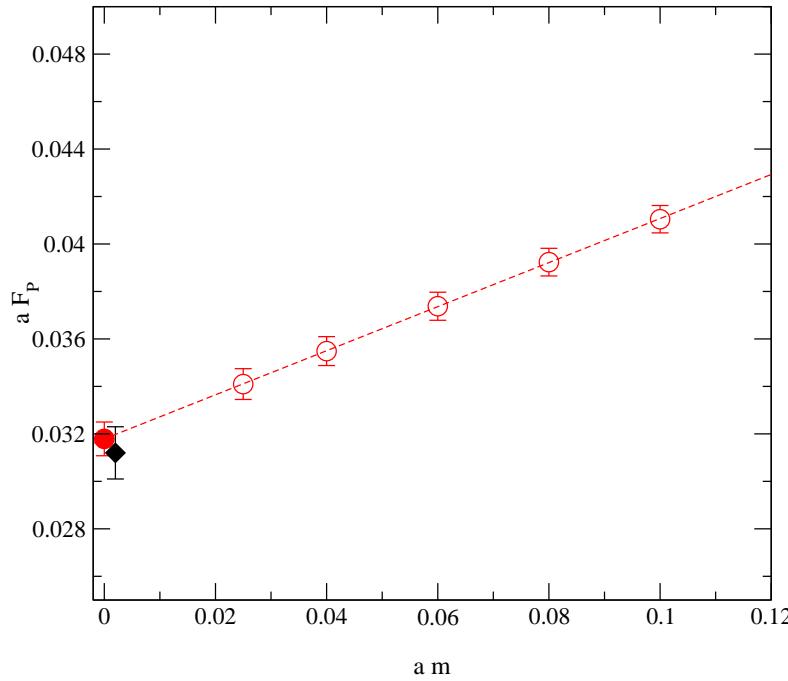
Giusti et al., $[p]$

Recent results in the ϵ -regime

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$$F \simeq 130 \text{ MeV}$$

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p-regime

Calculational rule in CHPT for isotropic finite box with periodic boundary conditions:

- the Lagrangian is the same as in infinite volume
- the propagators must be made periodic:

$$G_L(\vec{x}, t) = \sum_{\vec{n}} G_\infty(\vec{x} + \vec{n}L, t)$$

p-regime

Calculational rule in CHPT for isotropic finite box with periodic boundary conditions:

Examples:

Gasser and Leutwyler (88)

$$M_\pi(L) = M_\pi \left[1 + \frac{1}{2N_f} \xi g_1(\lambda) + O(\xi^2) \right]$$

$$F_\pi(L) = F_\pi \left[1 - \frac{N_f}{2} \xi g_1(\lambda) + O(\xi^2) \right]$$

with

$$\lambda = M_\pi L, \quad \xi = (M_\pi / 4\pi F_\pi)^2$$

$$g_1(\lambda) = \sum'_{\vec{n}} \int_0^\infty dz e^{-\frac{1}{z} - \frac{z}{4} \vec{n}^2 \lambda^2} = \sum_{\vec{n} \neq \vec{0}} G_\infty(\vec{x} + \vec{n}L, t) \Big|_{t=\vec{x}=0}$$

Finite volume effects in the p -regime

Foundations: Gasser and Leutwyler (87)

quenched CHPT: Sharpe, Bernard and Golterman (90's)

Recent applications:

- two-pion states Lin, Martinelli, Pallante, Sachrajda and Villadoro (03)
- F_K and B_K Becirevic and Villadoro (03)
- m_p QCDSF (03)
- m_N , μ_N and g_A Beane and Savage (03-04)
- f_B and B_B Arndt and Lin (04)
- m_p Koma and Koma (4 days ago)

Finite volume effects in the p -regime

Foundations: Gasser and Leutwyler (87)

quenched CHPT: Sharpe, Bernard and Golterman (90's)

Talks at this conference:

- M_π , F_π and $\langle r^2 \rangle_V$ R. Lewis
- Lüscher Formula for m_p Y. Koma
- m_p and g_A M. Goeckeler
- f_B and B_B D. Lin

Lüscher formula for M_π

Lüscher (86)

$$M_\pi(L) - M_\pi = -\frac{3}{16\pi^2 M_\pi L} \int_{-\infty}^{\infty} dy \ F(\mathrm{i}y) e^{-\sqrt{M_\pi^2 + y^2} L} + O(e^{-\overline{M}L})$$

where $F(\nu)$ is the physical forward $\pi\pi$ scattering amplitude

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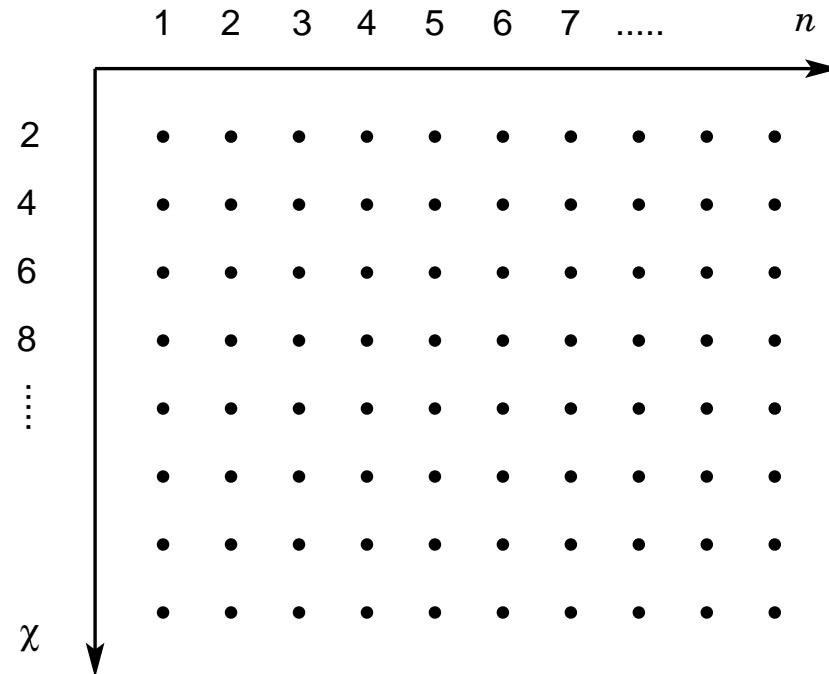
The formula gives the leading exponential term **exactly** to all orders in the chiral counting

Lüscher formula for M_π

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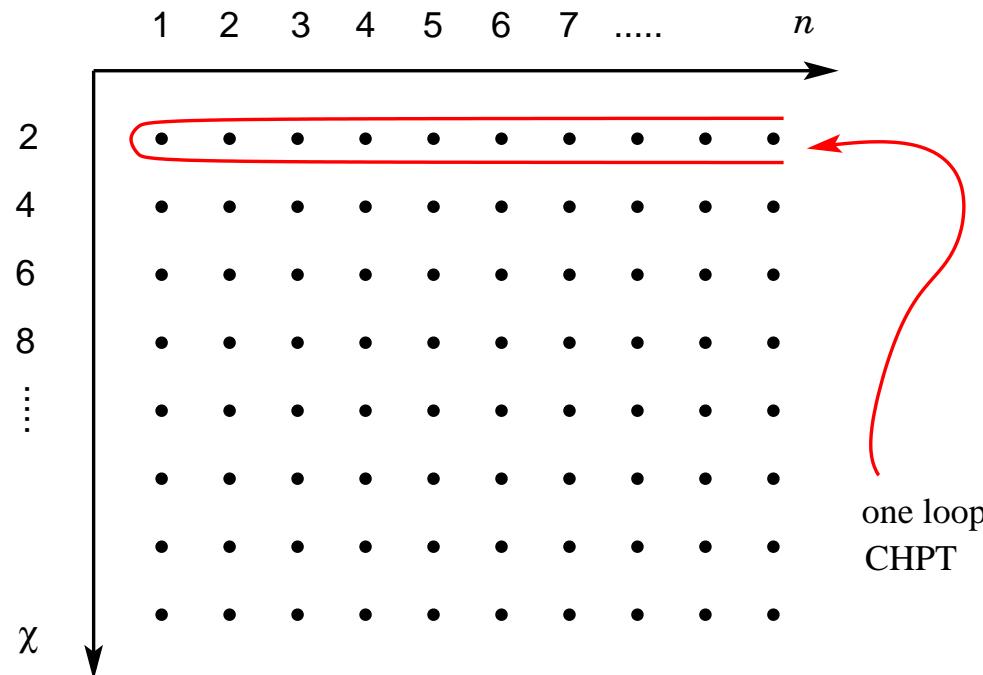


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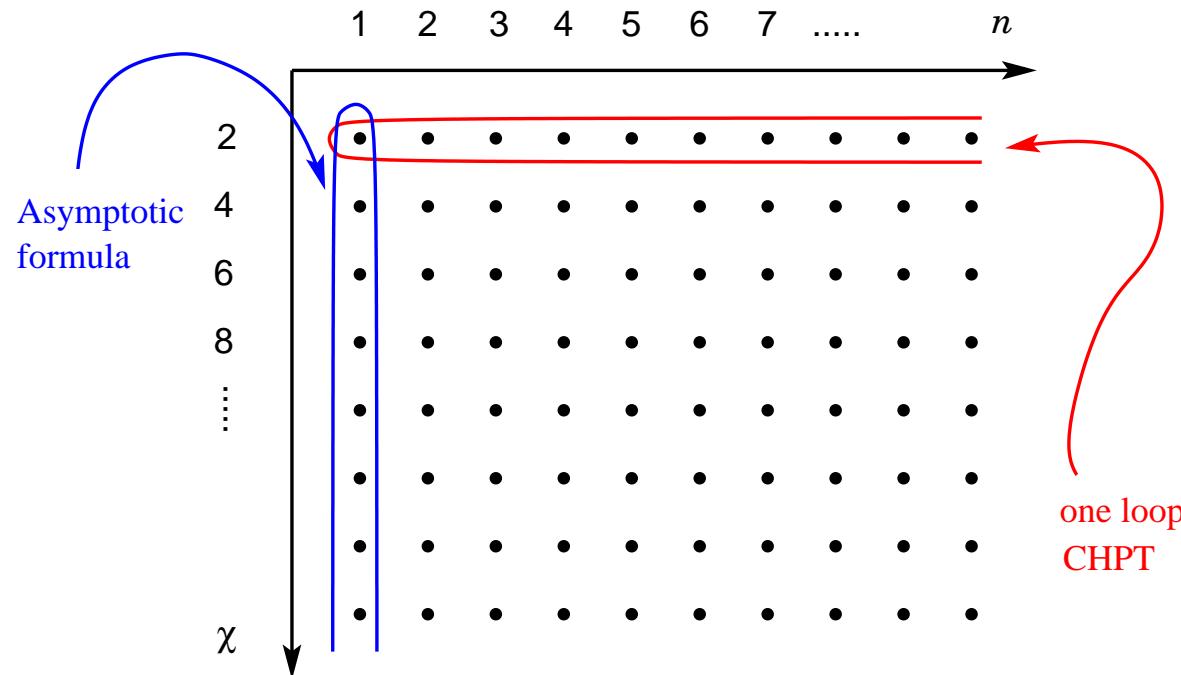


Lüscher formula for M_π

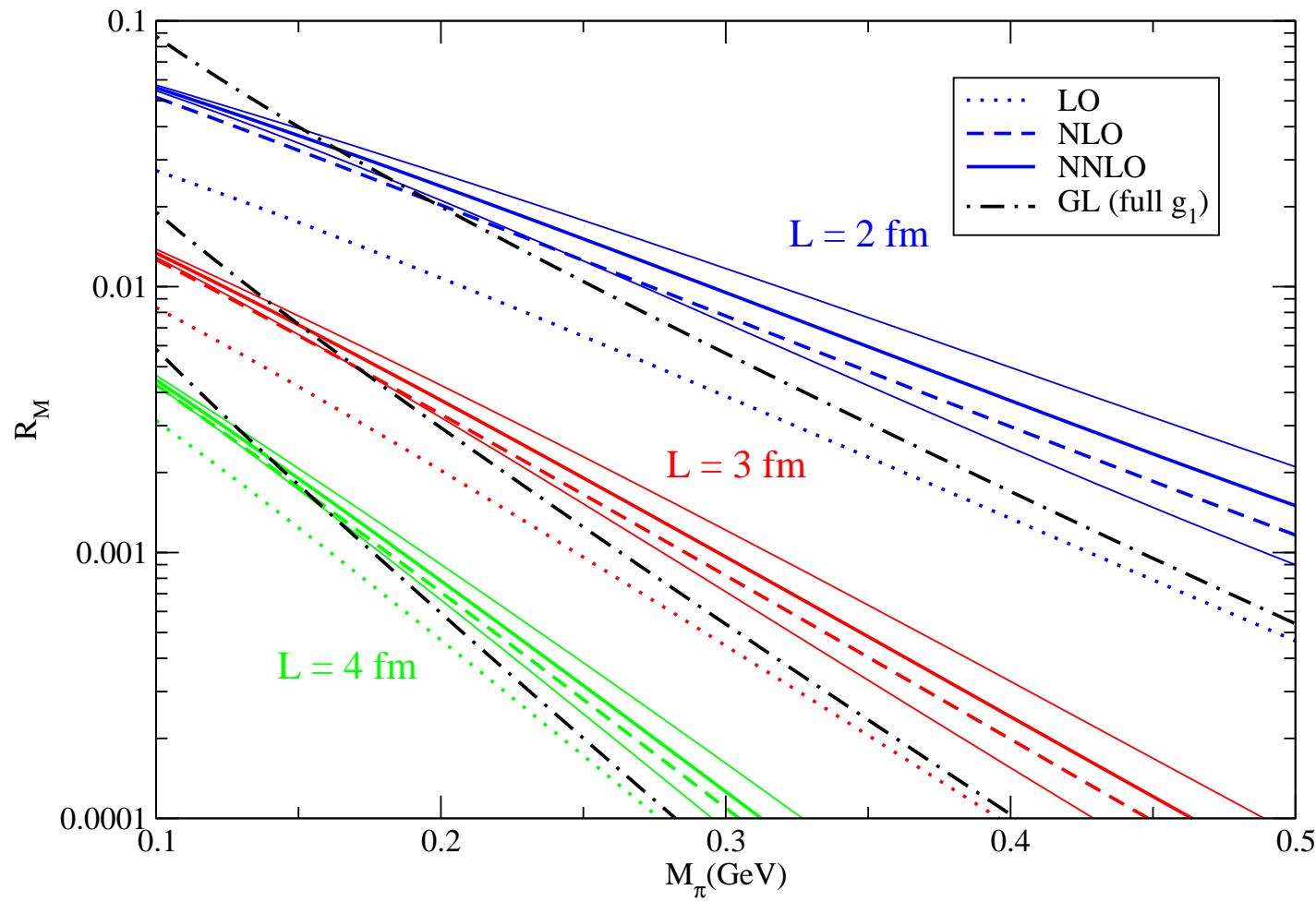
Lüscher (86)

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where $F(\nu)$ is the physical forward $\pi\pi$ scattering amplitude



Numerical study of $M_{\pi,L}$



G.C. and S. Dürr (04)

Extension of Lüscher's formula to all n 's

One-loop CHPT corrections proportional to

$$g_1(\lambda) = \sum_{\vec{n} \neq \vec{0}} G_\infty(\vec{x} + \vec{n}L, t) \Big|_{t=\vec{x}=0} = \sum_{|\vec{n}|=1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}| \lambda} K_1(|\vec{n}| \lambda)$$

where $m(|\vec{n}|)$ is the multiplicity with which a vector of length $|\vec{n}|$ occurs in 3-dim space

Extension of Lüscher's formula to all n 's

One-loop CHPT corrections proportional to

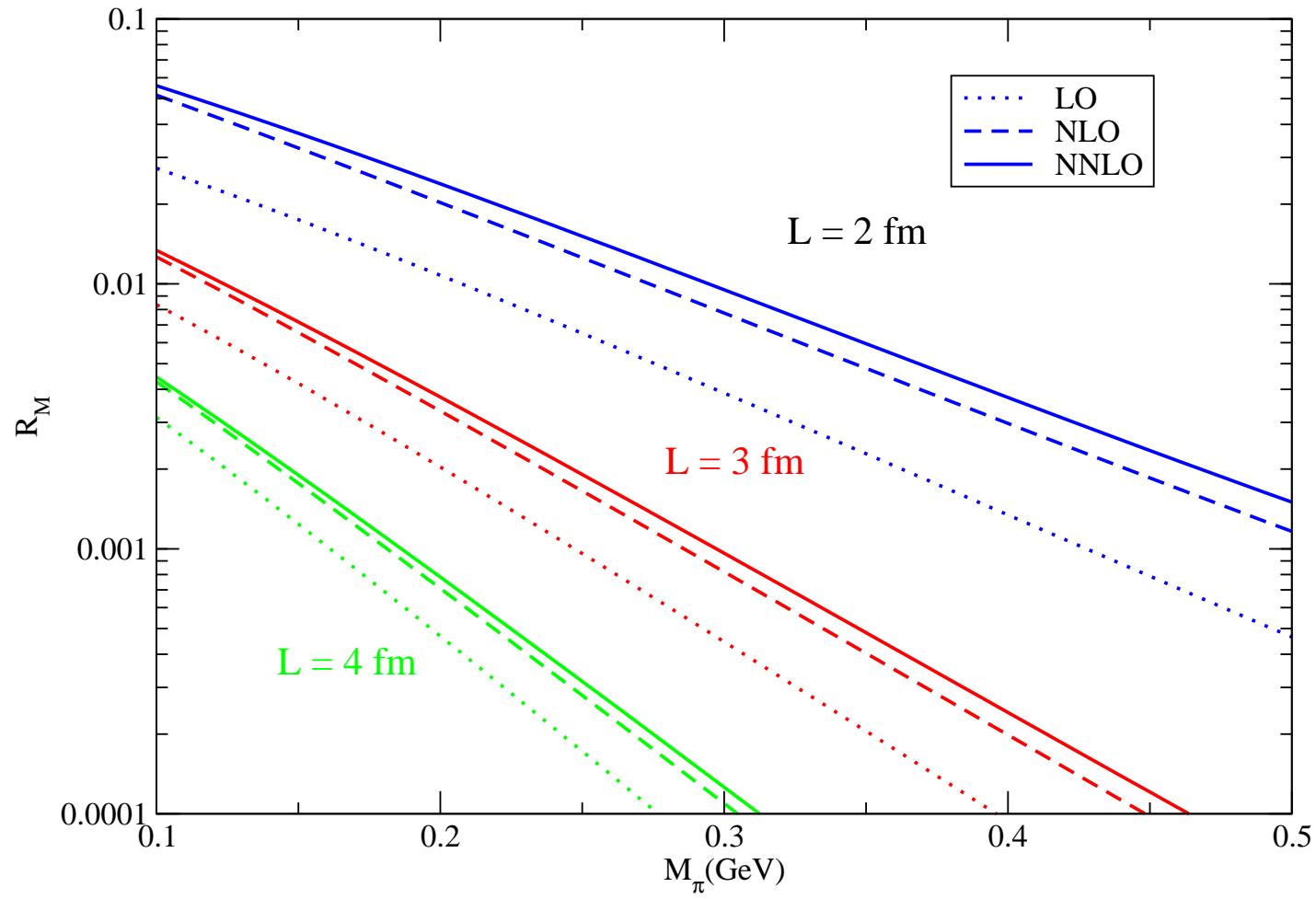
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Similarly one can extend Lüscher's formula to include the contributions with all $|\vec{n}|$ from a single propagator:

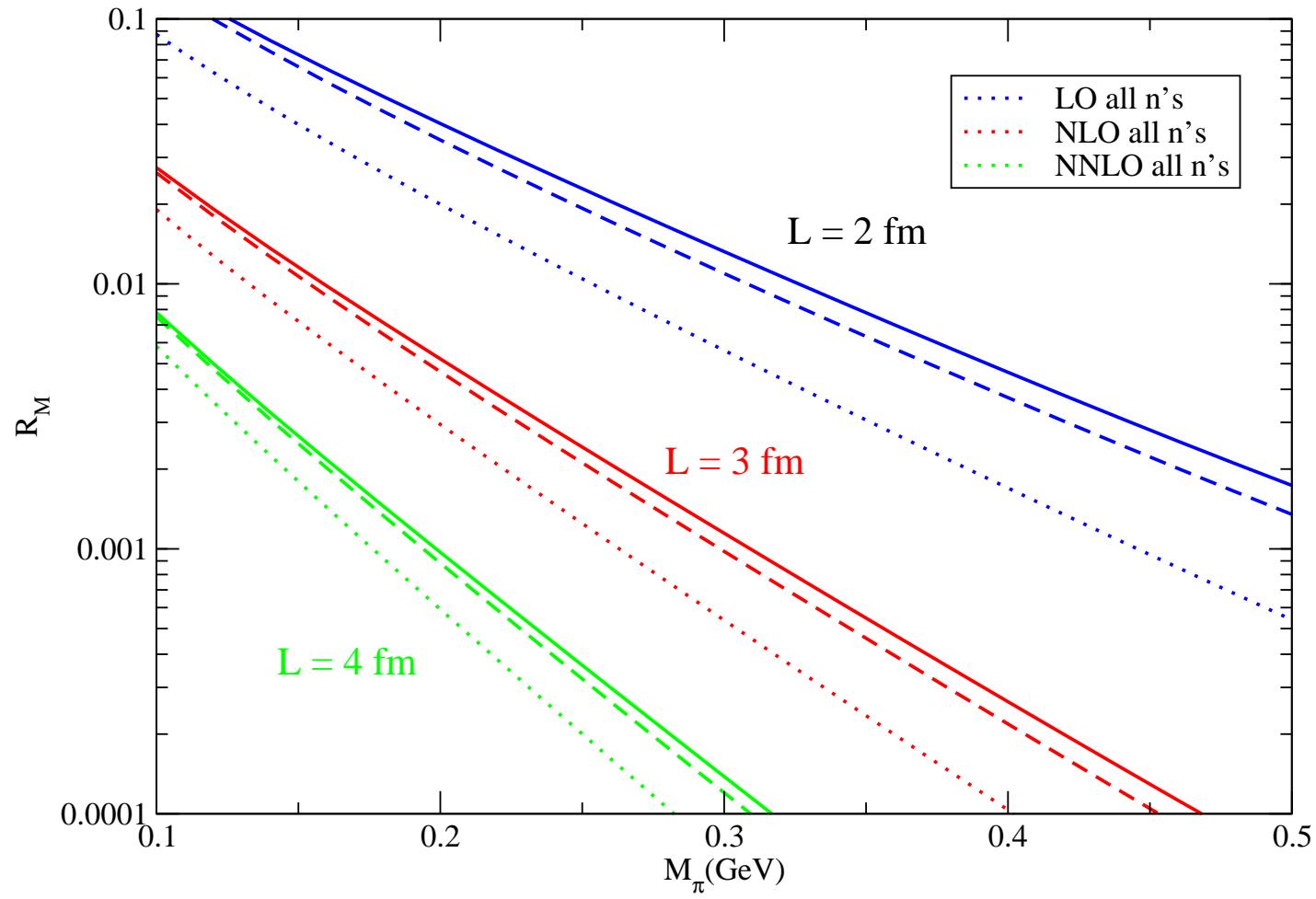
$$M_{\pi,L} - M_\pi = -\frac{1}{32\pi^2\lambda} \sum_{|\vec{n}|=1}^{\infty} \frac{m(|\vec{n}|)}{|\vec{n}|} \int_{-\infty}^{\infty} dy F(iy) e^{-\sqrt{\vec{n}^2(M_\pi^2+y^2)}L}$$

Not all exponentially subleading contributions are contained in this extended formula

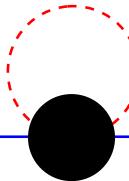
Subleading exponential terms in $M_{\pi,L}$



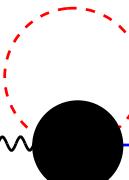
Subleading exponential terms in $M_{\pi,L}$



Extension to decay constants

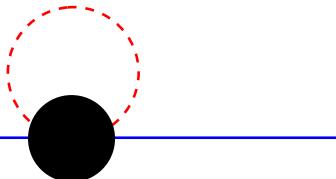


$$\Rightarrow \Delta M \propto \int_{-\infty}^{\infty} dy e^{-\sqrt{M_\pi^2+y^2}L} F(iy) + \dots$$
$$F(\nu) \Leftrightarrow \langle \pi\pi | T | \pi\pi \rangle$$

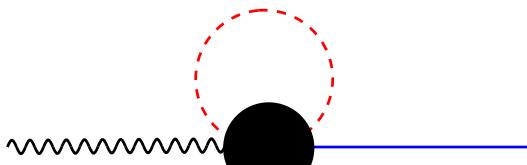


$$\Rightarrow \Delta F \propto \int_{-\infty}^{\infty} dy e^{-\sqrt{M_\pi^2+y^2}L} N(iy) + \dots$$
$$N(\nu) \Leftrightarrow \langle 0 | A_\mu | \pi\pi\pi \rangle \sim A(\tau \rightarrow 3\pi\nu_\tau)$$

Extension to decay constants



$$\Rightarrow \Delta M \propto \int_{-\infty}^{\infty} dy e^{-\sqrt{M_\pi^2+y^2}L} F(iy) + \dots$$
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The $\langle 0 | A_\mu | \pi\pi\pi \rangle$ amplitude must be subtracted:

$$N(\nu) = \langle (2\pi)_{I=0} \pi | A_\mu(0) | 0 \rangle - i Q_\mu \frac{F_\pi F(\nu)}{M_\pi^2 - Q^2}$$

G.C. und C. Haefeli 04

A Ward identity in finite volume

$$\langle 0 | A_\mu^i(0) | \pi^k(p) \rangle = i\delta^{ik} F_\pi p_\mu \quad \langle 0 | P^i(0) | \pi^k(p) \rangle = i\delta^{ik} G_\pi$$

Ward identity:

$$F_\pi M_\pi^2 = \hat{m} G_\pi$$

In finite volume, the Ward identity also holds, and implies:

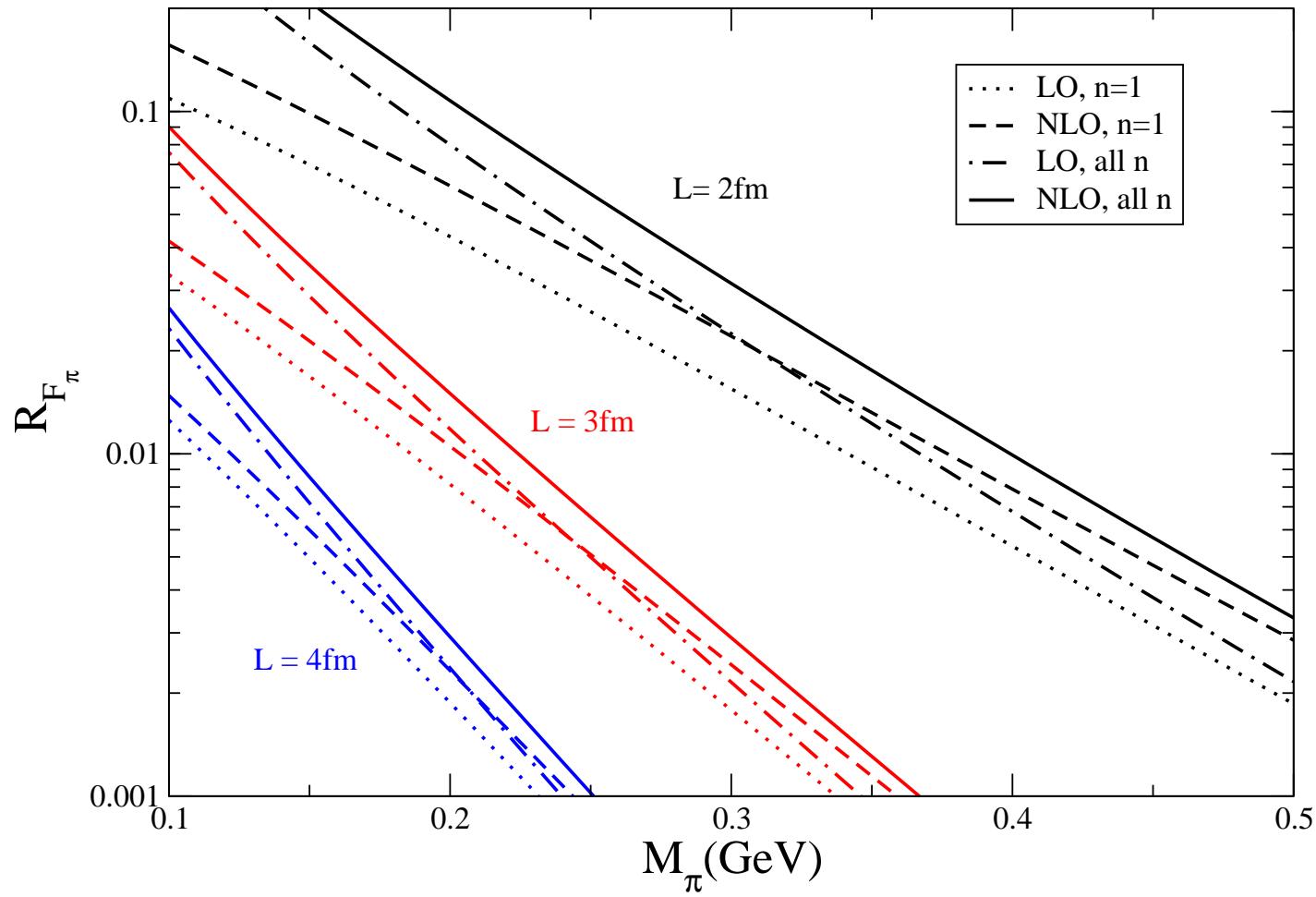
$$R_G = R_F + 2R_M \quad (R_X := \Delta X_\pi / X_\pi)$$

This identity must also be valid for the asymptotic formulae – and in particular for the integrands:

$$N_G(\nu) = N_F(\nu) - \frac{F_\pi}{M_\pi} F(\nu)$$

this coincides with a Ward identity for 4-point functions

Corrections for F_π



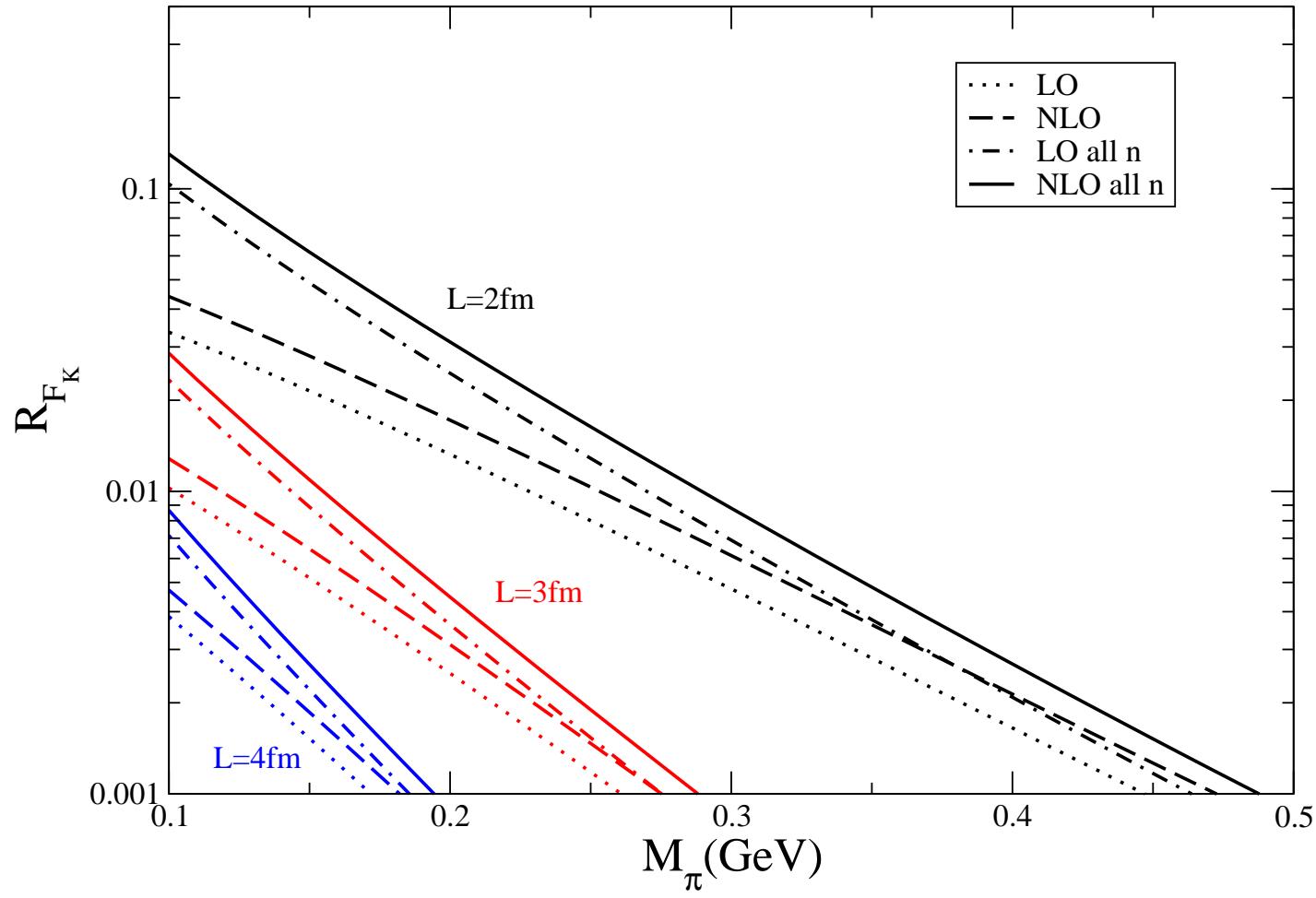
G.C. und C. Haefeli 04

Other applications

Quantity	Amplitude	Theory status
M_K	$A(\pi K \rightarrow \pi K)$	$O(p^6)$ (Bijnens et al.)
F_K	$A(K_{l4})$	$O(p^6)$ (Bijnens et al.)
M_η	$A(\pi\eta \rightarrow \pi\eta)$	$O(p^4)$ (Bernard et al.)
F_η	$A(\eta_{l4})$?
M_N	$A(\pi N \rightarrow \pi N)$	$O(p^4)$ various Authors
M_B	$A(\pi B \rightarrow \pi B)$?
F_B	$A(B_{l4})$?

Work in progress: G.C., S. Dürr, C. Haefeli, A. Fuhrer

Corrections for F_K



work in progress, G.C. und C. Haefeli

Summary

- For large volumes ($2LF_\pi \gg 1$), finite-volume effects can be calculated analytically within CHPT
- in the **ϵ -regime** ($M_\pi L < 1$) one can determine the LECs on the lattice **close to the chiral limit**
- remarkable recent work has shown that, although feasible, this is **technically challenging**
- several one-loop CHPT calculations in the **p -regime** ($M_\pi L \gg 1$) have appeared in the recent literature
- the combined use of CHPT and **asymptotic formulae à la Lüscher** offers the most efficient way to get to higher orders in CHPT

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- the combined use of CHPT and **asymptotic formulae à la Lüscher** offers the most efficient way to get to higher orders in CHPT
- **the extrapolation $L \rightarrow \infty$ can be made analytically!**